RINGS OF $SL_2(\mathbb{C})$-CHARACTERS AND THE KAUFFMAN BRACKET SKEIN MODULE

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Abstract. Let M be a compact orientable 3-manifold. The set of characters of $SL_2(\mathbb{C})$ representations of the fundamental group of M forms a closed affine algebraic set. We show that its coordinate ring is isomorphic to a specialization of the Kauffman bracket skein module modulo its nilradical. This is accomplished by making the module into a combinatorial analog of the ring, in which tools of skein theory are exploited to illuminate relations among characters. We conclude with an application, proving that a small manifold's specialized module is necessarily finite dimensional.

Keywords: knot, link, skein theory, representation theory, 3-manifold.

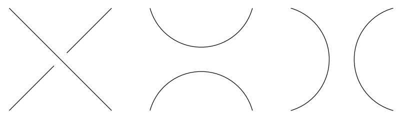
AMS (MOS) Subject Classification: 57M99.

1. Introduction

The Kauffman bracket skein module is an invariant of 3-manifolds which, until recently, was both difficult to compute and topologically mysterious. The discovery [3] that a specialization of the module dominates the ring of $SL_2(\mathbb{C})$ characters of the fundamental group shed some light on the meaning of the module. The relationship also provided estimates [2] [3] of the module's size and computational tools [4]. The central result of this paper sharpens the focus considerably, for we show that the specialization, modulo its nilradical, is exactly the ring of characters.

The construction depends upon a natural correspondence between knots and functions on the set of characters. Given an orientation, a knot corresponds to a conjugacy class in the fundamental group of a 3-manifold M. A formal linear combination of knots is therefore a template upon which one may evaluate characters of $\pi_1(M)$ represented in $SL_2(\mathbb{C})$. Conversely, one may interpret polynomial functions on the set of characters as linear combinations of links. These functions form an algebra, of which the the ring of characters is a quotient. The skein module is also a quotient (of the linear space of links) and it, too, has a ring structure. The correspondence between knots and functions descends to these quotients, where its kernel is exactly the nilpotent elements of the skein module.

The proof proceeds in three stages, the first of which (Section 2) recapitulates parts of [3] and [5]. We cover the necessary background, including precise definitions of the principal objects of study. Once the vocabulary is in place we define the map, Φ, taking knots to functions on the character set. The proof that it descends to a surjection on the quotients is quite simple, depending primarily on the following observation: the Kauffman bracket skein relation maps to the fundamental $SL_2(\mathbb{C})$



L L_0 L_∞

FIGURE 1.

trace identity,

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B).$$

The characterization of $\ker \Phi$, however, is significantly more involved.

We first study the correspondence for handlebodies and free groups, beginning in Section 3 with an investigation of trace identities. We define a map, Ψ, which sends functions on characters into the skein module. It turns out that $\ker \Phi$ contains exactly those trace identities which Ψ does not send to zero. The main result recalls work of Procesi [14] and Razmyslov [16], who classified a large family of trace identities on arbitrary matrix rings. We show that these identities, when restricted to $SL_2(\mathbb{C})$, map to zero. The skein module emerges here as a useful combinatorial tool. Although it is now possible to give purely algebraic proofs of the results in this section, they were all discovered by experimenting with the skein module. We have retained the geometric arguments for they serve to illustrate the interplay between the two theories and—once the reader has become comfortable with the skein moves—they make for shorter proofs.

Section 3 attains sufficient conditions for a trace identity to vanish in the skein module; Section 4 provides the finishing touch. We rely on a defining set of polynomials for the character set given in [8], the central results of which are reiterated in an effort to keep this paper self contained. Most of these polynomials turn out to be specialized Procesi identities, while the remaining few succumb to other tools from Section 3. It follows from a standard result of algebraic geometry that the only trace identities not vanishing in the skein module are nilpotent. It is then a small step to extend the result to arbitrary compact 3-manifolds.

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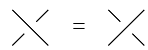
2. Definitions and Background

Let M be a compact orientable 3-manifold. The Kauffman bracket skein module of M is an algebraic invariant, denoted $K(M)$, which is built from the set $\mathcal{L}_M$ of framed links in M. By a framed link we mean an embedded collection of annuli considered up to isotopy in M, and we include the empty collection $\emptyset$. Three links L, L_0 and L_∞ are said to be *Kauffman skein related* if they can be embedded identically except in a ball where they appear as shown in Figure 1 (framings are vertical with respect to the page). The notation $L \amalg \bigcirc$ indicates the union of L with an unlinked 0-framed unknot.

Let R denote the ring of Laurent polynomials $\mathbb{C}[A^{\pm 1}]$ and $R\mathcal{L}_M$ the free R-module with basis $\mathcal{L}_M$. If L, L_0 and L_∞ are Kauffman skein related then $L -$

$AL_0 - A^{-1}L_\infty$ is called a *skein relation.* For any L in $\mathcal{L}_M$ the expression $L \amalg \bigcirc + (A^2 + A^{-2})L$ is called a *framing relation.* Let $S(M)$ be the smallest submodule of $R\mathcal{L}_M$ containing all possible skein and framing relations. We define $K(M)$ to be the quotient $R\mathcal{L}_M/S(M)$.

The indeterminate A is often interpreted as a complex number so that $K(M)$ becomes a vector space. It seems that the simplest value is $A = -1$, and we let $V(M)$ denote this specialization. Notice that the specialized skein relations imply



in $V(M)$. There is a product on links, $L_1L_2 = L_1 \cup L_2$, which makes $V(M)$ into a commutative algebra with $\emptyset$ serving as the identity. If follows from [1, Theorem 1] that $V(M)$ is generated by a finite set of knots.

By a *representation* we mean a homomorphism of groups

$$\rho : \pi_1(M) \to SL_2(\mathbb{C}).$$

The *character* of a representation is the composition

$$\chi_\rho = \text{trace} \circ \rho,$$

and $X(M)$ denotes the set of all characters. For each $\gamma \in \pi_1(M)$ there is a function $t_\gamma : X(M) \to \mathbb{C}$ given by $\chi_\rho \mapsto \chi_\rho(\gamma)$. The following theorem appears to have been discovered independently by Vogt [17] and Fricke [6], first proved by Horowitz [9], and then rediscovered by Culler and Shalen [5].

Theorem 1. (Vogt, Fricke, Horowitz, Culler–Shalen) *There exits a finite set of elements $\{\gamma_1, \dots, \gamma_m\}$ in $\pi_1(M)$ such that every t_γ is an element of the polynomial ring $\mathbb{C}[t_{\gamma_1}, \dots, t_{\gamma_m}]$.*

For Culler and Shalen, Theorem 1 was an initial step in a much deeper result.

Theorem 2. (Culler–Shalen) *If every t_γ is an element of $\mathbb{C}[t_{\gamma_1}, \dots, t_{\gamma_m}]$, then $X(M)$ is a closed algebraic subset of $\mathbb{C}^m$.*

Recall that a *closed algebraic set* X in $\mathbb{C}^m$ is the common zero set of an ideal of polynomials in $\mathbb{C}[x_1, \dots, x_m]$. The elements of $\mathbb{C}[x_1, \dots, x_m]$ are *polynomial functions* on X, and the functions x_i are *coordinates* on X. The quotient of $\mathbb{C}[x_1, \dots, x_m]$ by the ideal of polynomials vanishing on X is called the *coordinate ring* of X. Different choices of coordinates would clearly lead to different parameterizations of X, but it follows from [5] that any two parameterizations of $X(M)$ are equivalent via polynomial maps. Hence their coordinate rings are isomorphic and we may identify them as one object: the *ring of characters* of $\pi_1(M)$, which we denote by $\mathcal{R}(M)$.

Each knot K determines a unique t_γ as follows. Let $\vec{K}$ denote an unspecified orientation on K. Choose any $\gamma \in \pi_1(M)$ such that $\gamma \simeq \vec{K}$ (meaning the loop γ is freely homotopic to an embedding of $\vec{K}$). Since trace is invariant under conjugation it makes sense to define $\chi_\rho(\vec{K}) = \chi_\rho(\gamma)$. Since $\mathrm{tr}(A) = \mathrm{tr}(A^{-1})$ in $SL_2(\mathbb{C})$ we can also define $\chi_\rho(K) = \chi_\rho(\gamma)$. Thus K determines the map t_γ. Conversely, any t_γ is determined by some (non-unique) K. The main theorem of [3] is that this correspondence is well defined at the level of $V(M)$.

Theorem 3. *The map* $\Phi : V(M) \to \mathcal{R}(M)$ *given by*

$$\Phi(K)(\chi_\rho) = -\chi_\rho(K)$$

is a well defined surjective map of algebras. If $V(M)$ *is generated by the knots* $K_1, \dots, K_m$ *then* $-\Phi(K_1), \dots, -\Phi(K_m)$ *are coordinates on* $X(M)$.

Proof. Let $\mathbb{C}^{X(M)}$ denote the algebra of functions from $X(M)$ to $\mathbb{C}$. Define a map

$$\widetilde{\Phi} : \mathbb{C}\mathcal{L}_M \to \mathbb{C}^{X(M)}$$

as follows. If K is a knot set

$$\widetilde{\Phi}(K)(\chi_\rho) = -\chi_\rho(K).$$

If L is a link with components $K_1, \dots, K_n$ set

$$\widetilde{\Phi}(L) = \prod_{i=1}^{n} \widetilde{\Phi}(K_i).$$

Set $\widetilde{\Phi}(\emptyset) = 1$ and extend linearly.

Consider the image of $S(M)$ under $\widetilde{\Phi}$. For a framing relation, $L \amalg \bigcirc + 2L$, we have

$$\begin{aligned} \widetilde{\Phi}(L \amalg \bigcirc + 2L)(\chi_\rho) &= \widetilde{\Phi}(L)\widetilde{\Phi}(\bigcirc + 2\,\emptyset) \\ &= -\chi_\rho(\bigcirc) + 2 \\ &= -\mathrm{tr}(\mathrm{Id}) + 2 \\ &= 0. \end{aligned}$$

Next, let $L + L_0 + L_\infty$ be a skein relation in which L and L_0 are knots. It follows that L_∞ has two components, K_1 and K_2. Assume embeddings as in Figure 1 and choose a base point $*$ in the neighborhood where L, L_0 and L_∞ differ. It is now possible to find loops a and b in $\pi_1(M, *)$ so that a slight perturbation of ab gives $\vec{L}$. With favorable orientations on the other knots we have $ab^{-1} \simeq \vec{L}_0$, $a \simeq \vec{K}_1$, and $b \simeq \vec{K}_2$. Given any χ_ρ, set $A = \rho(a)$ and $B = \rho(b)$ so that

$$\begin{aligned} \widetilde{\Phi}(L + L_0 + L_\infty)(\chi_\rho) &= -\chi_\rho(L) - \chi_\rho(L_0) + \chi_\rho(K_1)\chi_\rho(K_2) \\ &= -\mathrm{tr}(AB) - \mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)\mathrm{tr}(B) \\ &= 0. \end{aligned}$$

Finally, note that every skein relation can be written as $L' \cup L + L' \cup L_0 + L' \cup L_\infty$ where L and L_0 are knots. Hence $\widetilde{\Phi}$ descends to a well defined map of algebras,

$$\Phi : V(M) \to \mathbb{C}^{X(M)},$$

which is determined by its values on knots.

Let $K_1, \dots, K_m$ be generators of $V(M)$. Every element of $V(M)$ can be written as a polynomial in these knots, so the image of Φ lies in $\mathbb{C}[-\Phi(K_1), \dots, -\Phi(K_m)]$. Since each t_γ is equal to $-\Phi(K)$ for some knot K, Theorems 1 and 2 imply that the functions $-\Phi(K_i)$ are coordinates on $X(M)$. It follows that Φ maps onto $\mathcal{R}(M)$. $\square$

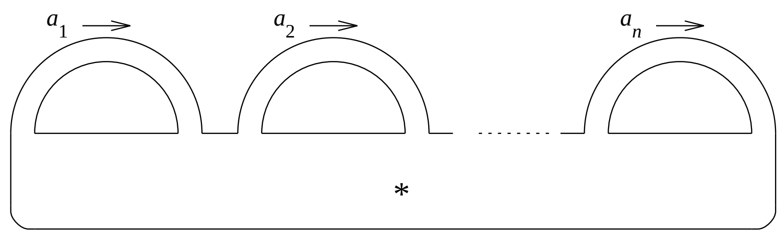


Figure 2.

3. Trace Identities

In the previous section we obtained a surjection $\Phi : V(M) \to \mathcal{R}(M)$ based on a natural correspondence between knots and functions on $X(M)$. Under this correspondence elements of $S(M)$ were sent to polynomials that vanish on $X(M)$, making Φ well defined. Our ultimate goal is to show that ker Φ is the set of nilpotent elements in $V(M)$. To this end we reverse the correspondence, mapping polynomials on $X(M)$ to elements of $V(M)$. For now, we will treat only the case where M is a handlebody. In this setting the kernel of Φ consists of polynomials that vanish on $X(M)$ but not in $V(M)$.

For the time being we will be concerned only with free groups, so throughout this section H will denote the manifold $P \times I$ where P is the planar surface in Figure 2. We also fix a base point $*$ in P and a set of generators $\{a_1, \dots, a_n\}$ for $\pi_1(H, *)$. Each loop a_i travels once across the i-th handle in the direction shown in Figure 2. Let $\mathcal{W}$ denote $\pi_1(H, *)$ modulo the equivalence

$$w \sim w' \iff w' = w^{-1} \text{ or } w' = gwg^{-1} \text{ for some } g \in \pi_1(H, *).$$

Consider the ring of polynomials $\mathbb{C}[\mathcal{W}]$.

Example 1.

$$p = (a_1)(a_2)(a_3) - (a_1a_2)(a_3) - (a_1a_3)(a_2) - (a_2a_3)(a_1) + (a_1a_2a_3) + (a_1a_3a_2)$$

Example 2.

$$\begin{aligned} q = {} & (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_1a_2)^2 + (a_1a_3)^2 + (a_2a_3)^2 \\ & + (a_1a_2a_3)^2 + (a_1a_2)(a_1a_3)(a_2a_3) + (a_1a_2a_3)(a_1)(a_2)(a_3) \\ & - (a_1a_2a_3)(a_1)(a_2a_3) - (a_1a_2a_3)(a_2)(a_1a_3) - (a_1a_2a_3)(a_3)(a_1a_2) \\ & - (a_1)(a_2)(a_1a_2) - (a_1)(a_3)(a_1a_3) - (a_2)(a_3)(a_2a_3) - 4 \end{aligned}$$

The parentheses are necessary to distinguish multiplication in $\pi_1(H)$ from multiplication in $\mathbb{C}[\mathcal{W}]$. Note that there is some ambiguity in the notation for an individual element of $\mathbb{C}[\mathcal{W}]$. For instance $(w^2) + (1) - (w)^2$ is the same as $(ww) + (ww^{-1}) - (w)(w^{-1})$. Occasionally it will be convenient to write a polynomial using non-reduced words.

A representation of $\pi_1(H, *)$ in $SL_2(\mathbb{C})$ is any assignment of matrices to each a_i. Letting parentheses denote the operation of trace, each element of $\mathbb{C}[\mathcal{W}]$ becomes a function from the representation space to $\mathbb{C}$. The elements of $\mathbb{C}[\mathcal{W}]$ that vanish as functions on the set of representations are called $SL_2(\mathbb{C})$ *trace identities*. They form an ideal $\mathcal{I} \subset \mathbb{C}[\mathcal{W}]$.

Each $w \in \mathcal{W}$ corresponds to a unique unoriented curve up to homotopy in H. We will use K_w to denote any knot in this homotopy class. Since crossings are irrelevant, K_w represents a unique element of $V(H)$. The assignment $w \mapsto -K_w$ defines a surjection of algebras,

$$\Psi : \mathbb{C}[\mathcal{W}] \to V(H).$$

The map Ψ turns an element of $\mathbb{C}[\mathcal{W}]$ into a linear combination of links in $V(H)$, where we can apply skein theory. The basic tool for calculating in $V(H)$ is a resolving tree. Let T be a finite, connected, contractible graph in which no vertex has valence greater than three. Assume that each vertex is labeled cL for some $c \in \mathbb{C}$ and some $L \in \mathcal{L}_H$. Assume further that there is a distinguished vertex c_0L_0 called the *root*. Define the potential of a vertex to be the number of edges in a path to the root. A (necessarily univalent) vertex that is not adjacent to one of higher potential is called a *leaf*. We say T is a *resolving tree* for c_0L_0 if each vertex cL satisfies exactly one of the following.

1. cL is a leaf.
2. cL is adjacent to exactly one higher potential vertex, $-2cL$.
3. cL is adjacent to exactly two higher potential vertices, $c'L'$ and $c''L''$, in which case $cL - c'L' - c''L''$ is a framing relation.

Figure 3, in which the dots represent a thrice punctured plane, is an example of a resolving tree for any knot that projects to the leftmost diagram. It is also an example of the most common way to produce a resolving tree. Beginning with a projection of the root, the tree grows by smoothing one crossing at a time. Once all crossings have been eliminated, trivial circles are removed via framing relations. Summing over the leaves gives the *standard resolution* of the root—an element of $\mathbb{C}\mathcal{L}_H$ which is equal to the root in $V(H)$. Although the procedure given here does not result in a unique tree, the following theorem [15] implies uniqueness of the standard resolution in $\mathbb{C}\mathcal{L}_H$.

Theorem 4. (Przytycki) *The links in H represented by diagrams in P with no crossings and no trivial circles are a basis for $V(H)$.*

A *resolving forest* for an element of $\mathbb{C}\mathcal{L}_H$ is simply a collection of trees, one for each term in the linear combination. As with individual links, there is a standard resolution of each element of $\mathbb{C}\mathcal{L}_H$. Summing the potential function over all vertices assigns a useful complexity to a forest, the *total potential*.

The remainder of this section is devoted to the establishment of conditions under which Ψ maps an identity to zero.

Lemma 1. *In Examples 1 and 2 we have $\Psi(p) = \Psi(q) = 0$.*

Proof. The root and leaves of the tree in Figure 3 are links representing the terms of p. One may check that the identity in $V(H)$ given by this resolution is precisely $\Psi(p)$. For q, resolve the diagram in Figure 4, which represents $-(a_1a_2)(a_1a_3)(a_2a_3)$. □

If p is a trace identity then a natural way to produce a new trace identity, q, is to substitute new words for each a_i in p. If $\Psi(p) = 0$ then one would hope $\Psi(q) = 0$ as well. Although this is true, the proof requires some effort.

Lemma 2. *Let $p \in \mathbb{C}[\mathcal{W}]$. If there exist words w_1 and w_2 such that $(w_1w_2) + (w_1w_2^{-1}) - (w_1)(w_2)$ divides p then $\Psi(p) = 0$. Also, if p is divisible by $(1) - 2$ then $\Psi(p) = 0$.*

Proof. In the first case consider the loop w_1w_2, but perturbed slightly so as to become an embedding. By definition the resulting knot is some $K_{w_1w_2}$. Similarly, perturb $w_1w_2^{-1}$, w_1 and w_2 to obtain embeddings of $K_{w_1w_2^{-1}}$, K_{w_1} and K_{w_2}. The perturbations may be chosen so that the embeddings of $K_{w_1w_2}$, $K_{w_1w_2^{-1}}$ and $K_{w_1}K_{w_2}$ coincide outside of a small neighborhood of the base point. Within that neighborhood they appear as in Figure 1, so they form a Kauffman skein triple.

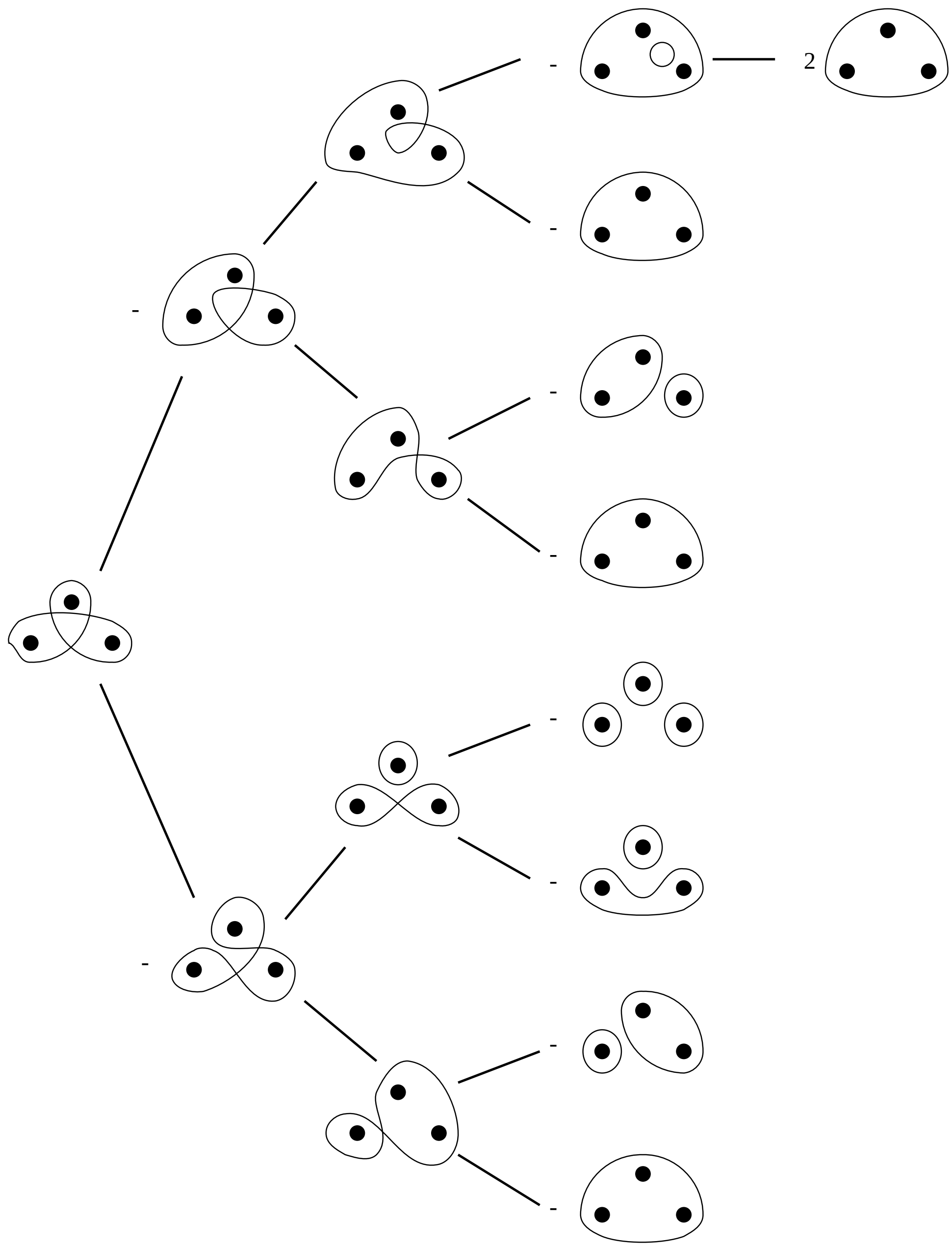


FIGURE 3.

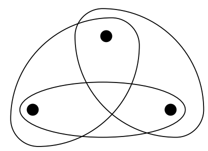


FIGURE 4.

We now have

$$
\begin{aligned}
0 &= -K_{w_1 w_2} - K_{w_1 w_2^{-1}} - K_{w_1} K_{w_2} \\
&= \Psi((w_1 w_2) + (w_1 w_2^{-1}) - (w_1)(w_2)),
\end{aligned}
$$

which implies $\Psi(p) = 0$. In the second case $\Psi(p)$ contains a factor of $\bigcirc + 2\,\emptyset$, which also implies $\Psi(p) = 0$. $\square$

Proposition 1. *Let $p \in \mathbb{C}[\mathcal{W}]$. Choose words $w_1, \dots, w_n$, and form a new polynomial q by substituting w_i for a_i in p. If $\Psi(p) = 0$ then $\Psi(q) = 0$.*

Proof. The proof is by induction on a complexity, $\kappa(p)$, which we define as follows. For each $w \in \mathcal{W}$ choose a diagram in P representing K_w. Express $\Psi(p)$ as an element of $\mathbb{C}\mathcal{L}_H$ using these diagrams, and then choose a forest for its standard resolution. Define $\kappa(p)$ to be the minimum total potential over all choices of diagrams and forests.

Assume first that $\kappa(p) = 0$, implying diagrams in which $\Psi(p)$ is expressed as its own standard resolution. If $\Psi(p) = 0$, we can invoke Theorem 4 to conclude that this particular expression of $\Psi(p)$ is formally zero in $\mathbb{C}\mathcal{L}_H$. It is not possible for a diagram to represent more than one w, so p (and hence q) must be identically zero.

Now assume that $\kappa(p) > 0$. Choose diagrams and a forest realizing $\kappa(p)$; also select a root cL which is not a leaf. There are three cases depending on the first resolution of cL.

Case 1: The resolution removes a self crossing of some component. Letting K denote that component, we construct loops in $\pi_1(H, *)$. Begin by choosing a point x near the crossing in question. Let α_0 be an arc running from $*$ to x; let α_1 be an arc running parallel to $\vec{K}$ until it returns to x; and let α_2 be an arc parallel to the remaining portion of $\vec{K}$. Set $\gamma_1 = \alpha_0 \alpha_1 \alpha_0^{-1}$ and $\gamma_2 = \alpha_0 \alpha_2 \alpha_0^{-1}$. We now have $K = K_{\gamma_1 \gamma_2}$. Furthermore, the resolution changes K into $K_{\gamma_1 \gamma_2^{-1}}$ and $K_{\gamma_1} K_{\gamma_2}$.

The term of p represented by cL must contain the indeterminate $(\gamma_1 \gamma_2)$. Replace that appearance of $(\gamma_1 \gamma_2)$ with $(\gamma_1)(\gamma_2) - (\gamma_1 \gamma_2^{-1})$, creating a new polynomial p'. Since $p - p'$ is divisible by $r = (\gamma_1 \gamma_2) + (\gamma_1 \gamma_2^{-1}) - (\gamma_1)(\gamma_2)$, Lemma 3 implies $\Psi(p') = 0$. Let q' and r' be the results of substituting w_i for a_i in p' and r (respectively). Removing the root cL from the forest for $\Psi(p)$ produces a forest for $\Psi(p')$ with lower total potential. Hence $\kappa(p') < \kappa(p)$ and, by induction, $\Psi(q') = 0$. Furthermore, r' has the form $(\gamma_1' \gamma_2') + (\gamma_1' {\gamma_2'}^{-1}) - (\gamma_1')(\gamma_2')$. Since r' divides $q - q'$ we have $\Psi(q) = 0$.

Case 2: The resolution removes a crossing between two components. In this case the components involved in the crossing correspond to loops γ_1 and γ_2, for which the resolution produces $K_{\gamma_1 \gamma_2}$ and $K_{\gamma_1 \gamma_2^{-1}}$. As in Case 1 we create p' by replacing $(\gamma_1)(\gamma_2)$ in p with $(\gamma_1 \gamma_2) + (\gamma_1 \gamma_2^{-1})$. The proof then proceeds by induction as before.

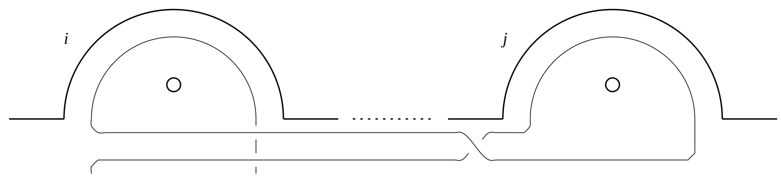


FIGURE 5. Band sum of $\Psi(a_i\alpha)$ and $\Psi(a_j)$.

Case 3: The resolution removes a trivial circle. The trivial circle corresponds to an appearance of (1) in p. Form p' by replacing that (1) with the scalar 2. Then create q and q' as above, noting that $(1) - 2$ divides both $p - p'$ and $q - q'$. As above, $\kappa(p') < \kappa(p)$, and it follows that $\Psi(q) = 0$. □

We would now like to consider a more general sort of trace identity. Let $\mathcal{S}_n$ denote the group of permutations of the set $\{a_1, \dots, a_n\}$. Let $\mathcal{S}_m$ denote the group of permutations of some subset $\{a_{i_1}, \dots, a_{i_m}\}$. Consider the group algebra $\mathbb{C}\mathcal{S}_m$. By writing the elements of $\mathcal{S}_m$ in cycle notation, including trivial cycles, we obtain expressions in $\mathbb{C}[\mathcal{W}]$. (Example 1, for instance.) In fact, since no inverses appear in these expressions, they can be regarded as functions on the set of m-tuples of 2×2 matrices. If an element of $\mathbb{C}\mathcal{S}_m$, regarded as such a function, vanishes for every assignment of 2×2 matrices we call it a *Procesi identity* on $\mathcal{S}_m$. Note that a Procesi identity is clearly an $SL_2(\mathbb{C})$ trace identity, but that the converse is just as clearly false.

Using the group algebra to encode Procesi identities is useful for the theorem we are about to prove, but there is a drawback as well. Multiplication in $\mathbb{C}\mathcal{S}_m$ is not the same as multiplication in $\mathbb{C}[\mathcal{W}]$. If p and q are elements of $\mathbb{C}\mathcal{S}_m$ we denote their product in the group algebra as $p \cdot q$, always assuming that p, q and $p \cdot q$ are written in cycle notation. Note that pq need not lie in $\mathbb{C}\mathcal{S}_m$, and that $p \cdot q$ may involve elements of $\mathcal{W}$ which do not appear in either p or q. Fortunately, the skein module keeps track of how multiplication in $\mathcal{S}_m$ rearranges the elements of $\mathcal{W}$.

Proposition 2. *Let $p \in \mathbb{C}\mathcal{S}_m$. If $\Psi(p) = 0$ then $\Psi(\tau \cdot p) = 0$ for every $\tau \in \mathcal{S}_n$.*

Proof. As an initial simplification assume that $\tau = (a_i a_j)$ with $i < j$, and that $\mathcal{S}_m$ permutes the set $\{a_1, \dots, a_m\}$. There are three cases, depending on the intersection of $\{a_1, \dots, a_m\}$ and $\{a_i, a_j\}$.

Case 1: $m < i$. As an element of $\mathbb{C}[\mathcal{W}]$, $\tau \cdot p$ factors into $(a_i a_j)p$. Hence $\Psi(p) = 0$ implies $\Psi(\tau \cdot p) = 0$.

Case 2: $i \le m < j$. Each term of p contains a cycle in which a_i appears. Assume that it is written $(a_i\alpha)$ and write $\tau \cdot (a_i\alpha)$ as $(a_j a_i \alpha)$. Fix a diagram for each term of $\Psi(p)$ with the property that it traverses handles 1 through m exactly once and misses the others. In a resolving forest for the standard resolution the skein relations take place in neighborhoods away from the handles, and no trivial circle runs once over a handle. Therefore every diagram in the forest meets the handles in precisely the same set of arcs, and we can apply the operation shown in Figure 5 to the entire forest. Note that this changes the diagram for $\Psi((a_i\alpha))$ into a diagram for $\Psi((a_j a_i \alpha))$, producing a resolution of $\Psi(\tau \cdot p)$. By Theorem 4, the resolution of $\Psi(p)$ is formally zero in $\mathbb{C}\mathcal{L}_H$. Since the resolution of $\Psi(\tau \cdot p)$ is obtained by applying Figure 5 to each term, it must also be zero.

Case 3: $j \leq m$. Each term of p contains either a cycle $(a_i \alpha a_j \beta)$ or a product of cycles $(a_i\alpha)(a_j\beta)$. The action of τ interchanges the two possibilities. Notice that the operation in Figure 6 interchanges the diagrams for $\Psi((a_i\alpha)(a_j\beta))$ and $\Psi((a_i\alpha a_j\beta))$. The proof then follows the resolving argument of Case 2.

Subject to our initial simplification, we now have $\Psi(\tau \cdot p) = 0$. Retaining the assumption that $\tau = (a_i a_j)$, we next allow $\mathcal{S}_m$ to permute any set $\{a_{i_1}, \dots, a_{i_m}\}$. A substitution converts this set into $\{a_1, \dots, a_m\}$, but preserves that fact that τ is a transposition. Hence, by Proposition 1, we again have $\Psi(\tau \cdot p) = 0$. Finally, since any element of $\mathcal{S}_n$ is a product of transpositions, $\Psi(\tau \cdot p) = 0$ for all $\tau \in \mathcal{S}_n$. ☐

Our interest in Procesi identities stems from a classification theorem due independently to Procesi [14] and Razmyslov [16]. Leron [12] is an excellent reference for the proof. For the sake of completeness we include some definitions taken from [7, Chapter 4]. A *Young diagram* for $\mathcal{S}_m$ is a collection of m boxes arranged in left justified rows of decreasing length. A *Young tableau* is an assignment of $a_{i_1}, \dots, a_{i_m}$ to the boxes. Figure 7 is an example of a Young diagram for S_{10} and a tableau using $\{a_1, \dots, a_{10}\}$. Given a tableau Y define P_Y to be the subgroup of $\mathcal{S}_m$ stabilizing the rows. For the tableau in Figure 7

$$P_Y \cong \mathcal{S}_3 \times \mathcal{S}_3 \times \mathcal{S}_2.$$

Similarly, define Q_Y to be the column stabilizer. The Young symmetrizer corresponding to Y is the element

$$\left(\sum_{\sigma \in P_Y} \sigma\right) \cdot \left(\sum_{\tau \in Q_Y} \operatorname{sgn}(\tau)\tau\right) \in \mathbb{C}\mathcal{S}_m.$$

Theorem 5. (Procesi, Razmyslov) *Procesi identities on a fixed $\mathcal{S}_m$ constitute an ideal in $\mathbb{C}\mathcal{S}_m$. The ideal is generated by Young symmetrizers corresponding to diagrams with at least three rows.*

Lemma 3. *Let Y be a Young tableau on $\{a_1, \dots, a_m\}$, and assume that a_m occupies the last box of a row and column as shown in Figure 8. Let Y' be the tableau obtained from Y by removing the box containing a_m. Using the notation $r_{s+1} = c_{t+1} = a_m$, we can express P_Y and Q_Y as the following disjoint unions:*

1. $P_Y = \bigcup_{i=1}^{s+1} (r_i a_m) \cdot P_{Y'}$, *and*
2. $Q_Y = \bigcup_{i=1}^{t+1} (c_i a_m) \cdot Q_{Y'}$.

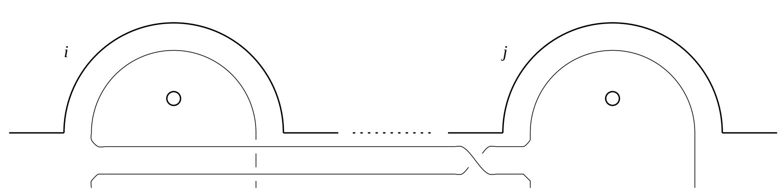


FIGURE 6. Band relating $\Psi((a_i\alpha)(a_j\beta))$ and $\Psi(a_i\alpha a_j\beta)$.

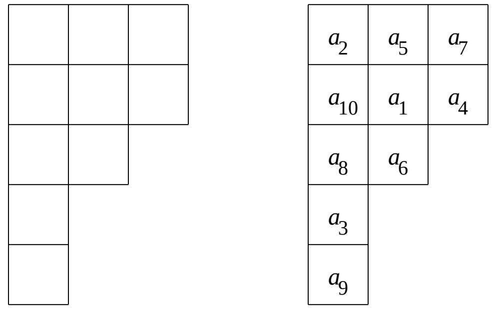


FIGURE 7.

Proof. Let $\lambda_1, \dots, \lambda_x$ be the lengths of the rows of Y'. A row stabilizer is a product of symmetric groups, so

$$|P_{Y'}| = \prod_{j=1}^{x} \lambda_j!, \quad \text{and}$$

$$|P_Y| = (s+1) \prod_{j=1}^{x} \lambda_j!.$$

Each coset $(r_i a_m) \cdot P_{Y'}$ stabilizes the rows of Y and, since each contains the element $(r_i a_m)$, they are disjoint. Counting elements finishes the proof for P_Y. The proof for Q_Y is similar. □

Theorem 6. *If p is a Procesi identity then $\Psi(p) = 0$.*

Proof. Implicit in the statement is the fact that p is a Procesi identity on some $\mathcal{S}_m$. We proceed by induction on m. By Theorem 5 and Proposition 2 we may assume that p is a Young symmetrizer corresponding to a digram with at least three rows. If $m = 3$ there is only one such diagram and p is the result of substituting a_{i_1}, a_{i_2} and a_{i_3} into Example 1. By Lemma 2 and Proposition 1, $\Psi(p) = 0$.

Now assume $m > 3$. Choose a diagram with at least three rows and a tableau Y satisfying the hypotheses of Lemma 3. A symmetrizer corresponding to any other tableau with the same diagram is obtained from this one by by a substitution. Therefore, by Proposition 1, it suffices to consider only Y. With notation as in

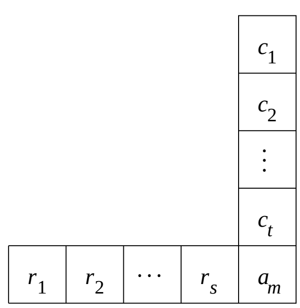


FIGURE 8.

Lemma 3, let p' be the symmetrizer corresponding to the tableau Y'. We then have

$$\begin{aligned}
p &= \left(\sum_{\sigma\in P_Y}\sigma\right)\cdot\left(\sum_{\tau\in Q_Y}\operatorname{sgn}(\tau)\tau\right)\\
&= \left[\sum_{i=1}^{s+1}\left(\sum_{\sigma\in P_{Y'}}(r_i a_m)\cdot\sigma\right)\right]\cdot\left[\sum_{j=1}^{t+1}\left(\sum_{\tau\in Q_{Y'}}\operatorname{sgn}((c_j a_m)\cdot\tau)(c_j a_m)\cdot\tau\right)\right]\\
&= \sum_{i,j}\operatorname{sgn}(c_j a_m)(r_i a_m)\cdot(c_j a_m)\cdot\left(\sum_{\sigma\in P_{Y'}}\sigma\right)\cdot\left(\sum_{\tau\in Q_{Y'}}\operatorname{sgn}(\tau)\tau\right)\\
&= \sum_{i,j}\operatorname{sgn}(c_j a_m)(r_i a_m)\cdot(c_j a_m)\cdot p'.
\end{aligned}$$

By induction we have $\Psi(p)=0$. □

4. The Coordinate Ring

In Section 3 we developed conditions under which Ψ carries an $SL_2(\mathbb{C})$ trace identity to zero. In this section we will complete the characterization of $\ker\Phi$, but to do so we must chose coordinates on $X(H)$.

Not only did Vogt [17], Fricke [6], and Culler and Shalen [5] apparently discover Theorem 1 independently, they all arrived at the same set of generators. Let $\gamma = a_{i_1}\cdots a_{i_m}$ be an element of G in which each a_{i_j} is distinct. Following [8] we adopt the shorthand notation $t_{i_1\cdots i_m}$ for the map t_γ. The generating set in all versions of Theorem 1 is $\mathcal{T} = \{t_{i_1\cdots i_m} \mid i_1 < i_2 < \cdots < i_m\}$.

Note that $\mathbb{C}[\mathcal{T}]$ becomes a subring of $\mathbb{C}[\mathcal{W}]$ by replacing $t_{i_1\cdots i_m}$ with $(a_{i_1}\cdots a_{i_m})$, so Ψ is well defined on $\mathbb{C}[\mathcal{T}]$. Theorem 1 says that for every $p\in\mathbb{C}[\mathcal{W}]$ there exists $q\in\mathbb{C}[\mathcal{T}]$ such that p and q represent the same element of $\mathcal{R}(H)$. We will need a stronger result, for which we turn to the combinatorial construction of $\mathcal{T}$ in [1].

Theorem 7. *Let $\mathcal{K}$ be a set of knots containing exactly one K_γ for each $t_\gamma\in\mathcal{T}$. Any link $L\in\mathcal{L}_H$ has a resolving tree whose leaves are monomials in $\mathbb{C}[\mathcal{K}]$.*

Corollary 1. *For every $p\in\mathbb{C}[\mathcal{W}]$ there exists $q\in\mathbb{C}[\mathcal{T}]$ such that $\Psi(p)=\Psi(q)$.*

The main result of [8] is the construction of an ideal, $\mathcal{J}_H$, which defines $X(H)$ in the coordinates $\mathcal{T}_0 = \{t_{i_1\cdots i_m}\in\mathcal{T} \mid m\le 3\}$. The radical of this ideal, $\sqrt{\mathcal{J}_H}$, is the ideal of trace identities in $\mathbb{C}[\mathcal{T}_0]$. The authors of [8] show that the trace identities in $\mathbb{C}[\mathcal{T}_0]$ generate those in $\mathbb{C}[\mathcal{T}]$, but we will need a slightly stronger result.

Lemma 4. (Compare [8, Lemma 4.1.1]). *Choose distinct indices $i,j,k,m_1,\dots,m_l$ and let $\alpha = m_1\cdots m_l$. If*

$$\begin{aligned}
q = &-2t_{ijk\alpha} + t_{ik}t_j t_\alpha - t_i t_j t_{k\alpha} - t_j t_k t_{\alpha i} - t_{ik}t_{j\alpha}\\
&+ t_{ij}t_{k\alpha} + t_{jk}t_{\alpha i} - t_{ikj}t_\alpha + t_i t_{jk\alpha} + t_j t_{k\alpha i} + t_k t_{\alpha ij}
\end{aligned}$$

then $\Psi(q)=0$.

Proof. Let p_{xyz} denote the Procesi identity of Example 1 with the substitutions $a_1=a_x$, $a_2=a_y$ and $a_3=a_z$. Consider the polynomial

$$p' = (a_1a_3)\cdot p_{234} - (a_3a_4)\cdot p_{124} - (a_1a_4)\cdot p_{123}$$

as an element of $\mathbb{C}[\mathcal{W}]$. By Theorem 5, p' is a Procesi identity, implying $\Psi(p') = 0$. Substituting $a_1 = a_i$, $a_2 = a_j$, $a_3 = a_k$ and $a_4 = a_{m_1} \cdots a_{m_l}$ in p' we obtain q, so Proposition 1 gives $\Psi(q) = 0$. $\square$

Proposition 3. *For every $q \in \mathbb{C}[\mathcal{T}]$ there exists $q_0 \in \mathbb{C}[\mathcal{T}_0]$ such that $\Psi(q) = \Psi(q_0)$.*

Proof. Let $q \in \mathbb{C}[\mathcal{T}]$. Define l to be the maximum length of a subscript appearing in q and let m be the number of maximum length subscripts. We say that the complexity of q is the ordered pair (l, m). The proof is by induction on complexity, ordered lexicographically. If $l \leq 3$ then q can be converted to $q_0 \in \mathbb{C}[\mathcal{T}_0]$ by repeated application of the identity in Example 1 (perhaps with a substitution of indices). The difference between any pair of successive stages is divisible by a Procesi identity, so Theorem 6 implies $\Psi(q) = \Psi(q_0)$.

If $l > 3$ then q contains some $t_{ijk\alpha}$ in which $ijk\alpha$ is a maximum length subscript. The identity of Lemma 4 allows us to replace $t_{ijk\alpha}$ with an expression involving only shorter subscripts. The result is a new polynomial q' with lower complexity and $\Psi(q) = \Psi(q')$. $\square$

We now state the main result of [8]. Define

$$\begin{aligned} M_{ii} &= t_i^2 - 4, \qquad \text{and} \\ M_{ij} = M_{ji} &= 2t_{ij} - t_i t_j, \qquad \text{if} \quad i < j. \end{aligned}$$

Theorem 8. (González-Acuña–Montesinos) *$X(M)$ is the zero set of the ideal $\mathcal{J}_H$ in $\mathbb{C}[\mathcal{T}_0]$ generated by the following polynomials.*

$$\begin{aligned} q_1 = {} & t_i^2 + t_j^2 + t_k^2 + t_{ij}^2 + t_{ik}^2 + t_{jk}^2 + t_{ijk}^2 + t_{ij}t_{ik}t_{jk} + t_{ijk}t_it_jt_k \\ & - t_{ijk}t_it_{jk} - t_{ijk}t_jt_{ik} - t_{ijk}t_kt_{ij} - t_it_jt_{ij} - t_it_kt_{ik} - t_jt_kt_{jk} - 4, \end{aligned}$$

in which i, j and k are distinct.

$$q_2 = \begin{vmatrix} M_{11} & M_{12} & M_{1i} & M_{1j} \\ M_{21} & M_{22} & M_{2i} & M_{2j} \\ M_{i1} & M_{i2} & M_{ii} & M_{ij} \\ M_{j1} & M_{j2} & M_{ji} & M_{jj} \end{vmatrix}, \quad \text{with } 2 < i < j \leq n.$$

$$q_3 = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{1i} \\ M_{21} & M_{22} & M_{23} & M_{2i} \\ M_{31} & M_{32} & M_{33} & M_{3i} \\ M_{j1} & M_{j2} & M_{j3} & M_{ji} \end{vmatrix}, \quad \text{with } 3 < i < j \leq n.$$

$$q_4 = (t_{123} - t_{132})(2t_{ijk} + t_it_jt_k - t_it_{jk} - t_jt_{ik} - t_kt_{ij}) - \begin{vmatrix} t_1 & t_{1i} & t_{1j} & t_{1k} \\ t_2 & t_{2i} & t_{2j} & t_{2k} \\ t_3 & t_{3i} & t_{3j} & t_{3k} \\ 2 & t_i & t_j & t_k \end{vmatrix},$$

in which $1 \leq i < j < k \leq n$ and t_{mm} denotes $t_m^2 - 2$.

The next step is to show that all of these polynomials lie in $\ker \Psi$. The proofs involved in this are closely modeled on those in [8]. Our contribution is the observation that q_2, q_3 and q_4 lift to Procesi identities, and that q_1 follows directly from a resolving tree. We will introduce further notation from [8] as it becomes necessary.

Lemma 5. $\Psi(q_1) = 0$.

Proof. Example 2, Lemma 1 and Proposition 1. ◻

Let $A_1, \dots, A_4$ be 2×2 matrices with

$$A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}.$$

Define

$$M(A_i) = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \end{pmatrix}, \ J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } J^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In general, (c_{ij}) will denote a 4×4 matrix and $|c_{ij}|$ its determinant. We use A^t to denote the transpose of A.

Lemma 6. (Compare [8, Lemma 4.6]). *The polynomial* $p = |2(x_i y_j) - (x_i)(y_j)|$ *is a Procesi identity on the symmetric group permuting* $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$.

Proof. Clearly p is an element of the group algebra over the permutations of $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$. To see that p is an identity, assign matrices A_i and B_j to each x_i and y_j. Direct calculation shows that

$$(\mathrm{tr}(A_i B_j)) = M(A_i) J M(B_j)^t,$$

and

$$(\mathrm{tr}(A_i B_j) - \mathrm{tr}(A_i)\mathrm{tr}(B_j)) = -M(A_i) J J^* M(B_j)^t.$$

Since $|I - J^*| = 0$ we have

$$|2\ \mathrm{tr}(A_i B_j) - \mathrm{tr}(A_i)\mathrm{tr}(B_j)| = |M(A_i) J (I - J^*) M(B_j)^t| = 0.$$

◻

Lemma 7. (Compare [8, Corollary 4.12]). *The polynomial*

$$\begin{aligned} p = {} & [(x_1 x_2 x_3) - (x_1 x_3 x_2)] \\ & \times [2(y_1 y_2 y_3) + (y_1)(y_2)(y_3) - (y_1)(y_2 y_3) - (y_2)(y_1 y_3) - (y_3)(y_1 y_2)] \\ & - \begin{vmatrix} (x_1) & (x_1 y_1) & (x_1 y_2) & (x_1 y_3) \\ (x_2) & (x_2 y_1) & (x_2 y_2) & (x_2 y_3) \\ (x_3) & (x_3 y_1) & (x_3 y_2) & (x_3 y_3) \\ 2 & (y_1) & (y_2) & (y_3) \end{vmatrix} \end{aligned}$$

is a Procesi identity on $\{x_1, x_2, x_3, y_1, y_2, y_3\}$.

Proof. (Compare [8, Lemma 4.10 and Proposition 4.11]). For any 2×2 matrices $A_1, \dots, A_4, B_1, \dots, B_4$ we have

$$\begin{aligned} |\mathrm{tr}(A_i B_j)| &= |M(A_i) J M(B_j)| \\ &= |J||M(A_i)||M(B_j)| \\ &= -|M(A_i)||M(B_j)|. \end{aligned}$$

If $A_4 = I$ then, by direct calculation,

$$|M(A_i)| = \mathrm{tr}(A_1 A_2 A_3) - \mathrm{tr}(A_1 A_3 A_2).$$

If $B_4 = I$ as well then

$$|\mathrm{tr}(A_i B_j)| = -[\mathrm{tr}(A_1 A_2 A_3) - \mathrm{tr}(A_1 A_3 A_2)][\mathrm{tr}(B_1 B_2 B_3) - \mathrm{tr}(B_1 B_3 B_2)].$$

Changing columns in $|\mathrm{tr}(A_i B_j)|$ and applying the identity of Example 1, we see that p vanishes for an arbitrary assignment of 2×2 matrices. As in Lemma 6, it is clearly an element of the appropriate group algebra. $\square$

Lemma 8. (Compare [8, Proposition 4.8]). $\Psi(q_2) = 0$.

Proof. Create $p \in \mathbb{C}[\mathcal{W}]$ by specializing the Procesi identity of Lemma 6 at $x_1 = y_1 = a_1$, $x_2 = y_2 = a_2$, $x_3 = y_3 = a_i$ and $x_4 = y_4 = a_j$. Theorem 6 and Proposition 1 imply $\Psi(p) = 0$. Note that p and q_2 are determinants of matrices that differ only along their diagonals. The differences between diagonal terms are of the form

$$2(a_m^2) - (a_m)^2 - t_m^2 + 4,$$

which can be rewritten as

$$2(a_m^2) - 2(a_m)^2 + 2(1) - 2(1) + 4.$$

Hence, q_2 may be obtained from p by a finite sequence of substitutions of the form $(a_m^2) = (a_m)^2 - (1)$ or $(1) = 2$. Each step involves a pair of polynomials whose difference is divisible either by $(a_m a_m) + (a_m a_m^{-1}) - (a_m)(a_m^{-1})$ or by $(1) - 2$, so Lemma 2 implies $\Psi(q_2) = \Psi(p) = 0$. $\square$

Lemma 9. (Compare [8, Proposition 4.9]). $\Psi(q_3) = 0$.

Proof. Specialize the identity of Lemma 6 at $x_1 = y_1 = a_1$, $x_2 = y_2 = a_2$, $x_3 = y_3 = a_3$, $x_4 = a_i$ and $y_4 = a_j$. Then proceed as in Lemma 8. $\square$

Lemma 10. (Compare [8, Corollary 4.12]). $\Psi(q_4) = 0$.

Proof. Specialize the Procesi identity of Lemma 7 at $y_1 = a_i$, $y_2 = a_j$, $y_3 = a_k$ and $x_m = a_m$ for $m = 1, 2, 3$. If $i > 3$ this is precisely q. If not, then proceed as in Lemmas 8 and 9, using the fact that t_{mm} denotes $t_m^2 - 2$. $\square$

This is enough to prove our claims about $\Phi : V(M) \to \mathcal{R}(M)$, but we may as well consider an arbitrary, compact, orientable 3-manifold. Let M be the result of adding 2-handles to H along curves $\{c_1, \ldots, c_m\}$ in ∂H. Choose words w_i in $\pi_1(H)$ so that, as a loop, each w_i is freely homotopic to some orientation of c_i. For each i and j form the polynomial $p_{ij} = (w_i a_j) - (a_j) \in \mathcal{R}(H)$. Using the obvious identification $\mathcal{R}(H) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_H}$, create an ideal $\mathcal{J}_M$ in $\mathbb{C}[\mathcal{T}_0]$ generated by $\mathcal{J}_H \cup \{p_{ij}\}$.

Theorem 9. (González-Acuña–Montesinos) *$X(M)$ is the zero set of $\mathcal{J}_M$ in $\mathbb{C}[\mathcal{T}_0]$.*

It follows immediately that $\mathcal{R}(M) = \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}$. We know that Φ maps $V(M)$ onto $\mathcal{R}(M)$ and it is clear that Ψ maps $\mathbb{C}[\mathcal{W}]$ onto $V(H)$, and hence onto $V(M)$. Using these maps, we can now see how $V(M)$ compares to $\mathcal{R}(M)$. From now on, consider Ψ to be the restriction to $\mathbb{C}[\mathcal{T}_0]$.

Proposition 4. $\mathcal{J}_M \subset \ker \Psi \subset \sqrt{\mathcal{J}_M}$

Proof. That $\mathcal{J}_H$ lies in $\ker \Psi$ is the content of Lemmas 5, 8, 9, and 10. To see that $\Psi(p_{ij}) = 0$, construct a knot K_{a_j} for each generator a_j. For each i and j, there is a band sum $c_i \#_b K_{a_j}$ producing a knot $K_{w_i a_j}$. Since $c_i \#_b K_{a_j} \cong K_{a_j}$ in M, we have $\Psi(p_{ij}) = K_{a_j} - K_{w_i a_j} = 0$.

For the second containment, note that Corollary 1 and Proposition 3 imply that $\Psi|_{\mathbb{C}[\mathcal{T}_0]}$ is still onto. It should now be clear that

$$\mathbb{C}[\mathcal{T}_0] \xrightarrow{\Psi} V(M) \xrightarrow{\Phi} \mathcal{R}(M) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}$$

is the canonical projection. □

There are various equivalent ways of phrasing the immediate implication of Proposition 4.

Theorem 10. *Let M be a compact orientable 3-manifold with Φ, Ψ, $\mathcal{J}_M$, and $\mathcal{T}_0$ defined as above. Denote the ideal of nilpotents in $V(M)$ by $\sqrt{0}$.*

1. *$X(M)$ is the zero set of $\ker \Psi$ in $\mathbb{C}[\mathcal{T}_0]$.*
2. *$\sqrt{\ker \Psi} = \sqrt{\mathcal{J}_M}$.*
3. *$\ker \Phi = \sqrt{0}$.*
4. *Φ induces an isomorphism $\widehat{\Phi} : V(M)/\sqrt{0} \to \mathcal{R}(M)$.*
5. *Ψ induces an isomorphism $\widehat{\Psi} : \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M} \to V(M)/\sqrt{0}$.*
6. *Under the identification of $\mathcal{R}(M)$ with $\mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}$, the maps $\widehat{\Psi}$ and $\widehat{\Phi}$ are inverses.*

Proof.

1. This is immediate from Proposition 4 and the fact the $X(M)$ is the zero set of both $\mathcal{J}_M$ and $\sqrt{\mathcal{J}_M}$.
2. Nullstellensatz.
3. Since $\mathcal{R}$ cannot, by definition, contain a non-zero nilpotent element, $\Phi(\sqrt{0}) = 0$. Suppose now that $\Phi(\alpha) = 0$, and write α as $\Psi(\beta)$. We have seen that
$$\mathbb{C}[\mathcal{T}_0] \xrightarrow{\Psi} V(M) \xrightarrow{\Phi} \mathcal{R}(M) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}$$
is the canonical projection. Hence, $\beta \in \sqrt{\mathcal{J}_M}$. It follows from Theorem 10 part (2) that $\Psi(\beta^n) = 0$ for some n, meaning α is nilpotent.
4. Theorem 3 and Theorem 10 part (3).
5. The composition
$$\mathbb{C}[\mathcal{T}_0] \xrightarrow{\Psi} V(M) \xrightarrow{\pi} V(M)/\sqrt{0} \xrightarrow{\widehat{\Phi}} \mathcal{R}(M) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}.$$
is also the canonical projection (here π is projection as well). Hence, $\ker \pi \circ \Phi = \sqrt{\mathcal{J}_M}$.
6. It is easy to see that both
$$\mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M} \xrightarrow{\widehat{\Psi}} V(M)/\sqrt{0} \xrightarrow{\widehat{\Phi}} \mathcal{R}(M) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}$$
and
$$V(M)/\sqrt{0} \xrightarrow{\widehat{\Phi}} \mathcal{R}(M) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M} \xrightarrow{\widehat{\Psi}} V(M)/\sqrt{0}$$
are the identity.

□

We conclude with an application. The author would like to thank Charles Frohman for suggesting that this result might follow quickly, Victor Camillo for encouraging us to disregard nilpotents, and Bernadette Mullins for pointing out the result from ring theory used in the proof. Recall that a 3-manifold is *small* if it contains no incompressible, non-boundary parallel surface.

Theorem 11. (Compare [2, Corollary 1]). *If M is small then* $\dim V(M) < \infty$.

Proof. Suppose that $X(M)$ has positive dimension. If follows that some component of $X(M)$ contains a curve whose smooth projective resolution contains an ideal point. From [5, 2.2.1] we then have a non-trivial splitting of $\pi_1(M)$, meaning M is not small. Hence, $X(M)$ consists of a finite set of points and $\mathcal{R}(M)$ is finite dimensional as a vector space. It is a standard result of commutative algebra that an ideal in a Noetherian ring contains some power of its radical. Thus, from Theorem 10 part (2), we obtain

$$\left(\sqrt{\mathcal{J}_M}\right)^n \subset \ker \Psi \subset \sqrt{\mathcal{J}_M}$$

for some n. Since $\mathcal{R}(M) \cong \mathbb{C}[\mathcal{T}_0]/\sqrt{\mathcal{J}_M}$, it is a simple exercise to show that $\mathbb{C}[\mathcal{T}_0]/\left(\sqrt{\mathcal{J}_M}\right)^n$ is finite dimensional. The result now follows from the fact that $V(M) \cong \mathbb{C}[\mathcal{T}_0]/\ker\Psi$, which in turn is the homomorphic image of $\mathbb{C}[\mathcal{T}_0]/\left(\sqrt{\mathcal{J}_M}\right)^n$. □

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